# Dynamic predictions from Joint Models using Super Learning

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# 1 Background & Motivation

Setting: Follow-up studies

- multiple longitudinal outcomes
  - biomarkers
  - patient parameters
  - patient reported outcome scores

- one or multiple endpoints
  - relapse of disease
  - requirement for intervention
  - death

# 1 Background & Motivation (cont'd)

**Obtain accurate predictions for the (cumulative) risk of an event to guide decision making**

**Using the available longitudinal information**

# 1 Background & Motivation (cont'd)

## University of Michigan Prostatectomy Data

- 3634 PCa patients followed-up in 1996–2013
  - aged 40 to 84 years with clinically localized cT1 to cT3 disease
  - received radical prostatectomy

# 1 Background & Aim (cont'd)

University of Michigan Prostatectomy Data

**Patients remain at risk of metastasis**

- Follow-up
  - * PSA levels at frequent intervals
  - * when PSA increases, physicians consider Salvage Therapy (ST)
  - * ST androgen deprivation therapy, radiation therapy, chemotherapy, and combinations

# 1 Background & Motivation (cont'd)

University of Michigan Prostatectomy Data

**Use the longitudinal PSA & baseline covariates to predict the risk of metastasis**

# 1 Background & Motivation (cont'd)

- Two main frameworks to obtain such predictions

  ▷ *Landmarking*

  * a series of Cox models at different landmark times
  * biomarker last value as a baseline covariate or a mixed model
  * Breslow estimator of survival probabilities

  ▷ *Joint Models*

  * complete specification of the joint distribution of the outcomes
  * direct derivation of conditional risk probabilities

# 1 Background & Motivation (cont'd)

## Landmarking

- ▷ *Advantages*
  - * easier to use, available in standard software
  - * can generalize to multiple biomarkers without (much) extra computational cost

- ▷ *Disadvantages*
  - * predictions not consistent
  - * not plausible LOCF for biomarkers
  - * does not account for measurement error and endogeneity
  - * not valid causal interpretation

# 1 Background & Motivation (cont'd)

## Joint Models

- ▷ *Advantages*
  - * consistent predictions
  - * accounts for measurement error and endogeneity
  - * biomarkers follow a trajectory
  - * valid causal interpretation

- ▷ *Disadvantages*
  - * computationally intensive
  - * *sensitive to modeling assumptions*

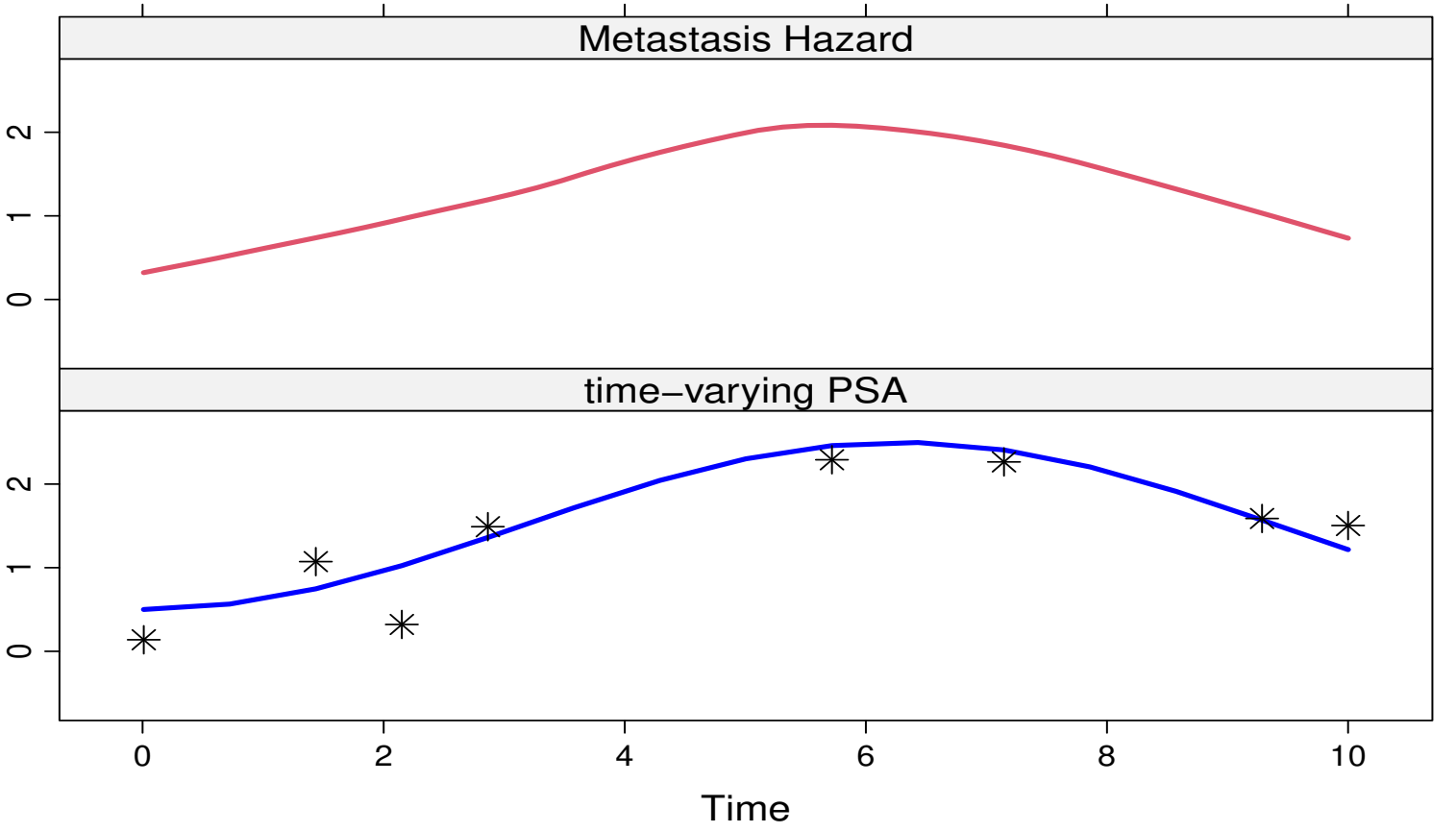
# 1 Background & Motivation (cont'd)

- *Sensitive to modeling assumptions*
  - ▷ *Longitudinal profiles shape*
    - * non-linear subject-specific trajectories
  - ▷ *Functional form*
    - * how to link the hazard of the event with the longitudinal outcome

## Joint Models Framework - Basic Idea

- ▷ Use a model to describe the subject-specific longitudinal trajectories
- ▷ Use these trajectories in a hazard model for the event
- ▷ Random effects explain the association

# 2 Joint Models (cont'd)



# 2 Joint Models (cont'd)

Some notation

- ▷ $T_i^*$: True event time for patient $i$
- ▷ $T_i$: Observed event time for patient $i$
- ▷ $\delta_i$: Event indicator, i.e., equals 1 for true events
- ▷ $y_i$: Longitudinal covariate

# 2 Joint Models (cont'd)

More formally

$$
\begin{cases}
h_i(t \mid \mathcal{H}_i(t, \boldsymbol{b}_i)) = h_0(t) \exp\{\boldsymbol{\gamma}^\top \boldsymbol{w}_i + f(\alpha, \mathcal{H}_i(t, \boldsymbol{b}_i))\}, \\
\qquad\qquad\qquad\qquad \mathcal{H}_i(t, \boldsymbol{b}_i) = \{\eta_i(s, \boldsymbol{b}_i); 0 \leq s \leq t\} \\
\\
y_i(t) = \eta_i(t, \boldsymbol{b}_i) + \varepsilon_i(t) \\
\qquad\quad = \boldsymbol{x}_i^\top(t)\boldsymbol{\beta} + \boldsymbol{z}_i^\top(t)\boldsymbol{b}_i + \varepsilon_i(t), \quad \varepsilon_i(t) \sim \mathcal{N}(0, \sigma^2), \\
\\
\boldsymbol{b}_i \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{D})
\end{cases}
$$

# 2 Joint Models (cont'd)

- We follow a Bayesian estimation paradigm treating $\theta$ and $\{b_i, i = 1, \ldots, n\}$ are regarded as parameters

- Inference is based on the full posterior distribution

$$p(\theta, b \mid T, \delta, y) = \frac{\prod_i p(T_i, \delta_i \mid b_i, \theta)\; p(y_i \mid b_i, \theta)\; p(b_i, \theta)\; p(\theta)}{\prod_i p(T_i, \delta_i, y_i)}$$

$$\propto \prod_{i=1}^{n} \Big\{ p(T_i, \delta_i \mid b_i, \theta)\; p(y_i \mid b_i, \theta)\; p(b_i, \theta) \Big\}\; p(\theta)$$

# 2 Joint Models (cont'd)

- Dynamic predictions from joint models

$$\pi_i(u \mid t) = \mathsf{Pr}\{T_i^* \leq u \mid T_i^* > t, \mathcal{Y}_i(t), \mathcal{D}_n\}, \quad u > t,$$

where

- $\mathcal{Y}_i(t) = \{y_i(s), 0 \leq s \leq t\}$ available measurements up to $t$
- $\mathcal{D}_n$ the sample used to fit the model

# 2 Joint Models (cont'd)

- Under the Bayesian formulation $\pi_i(u \mid t)$ is written as

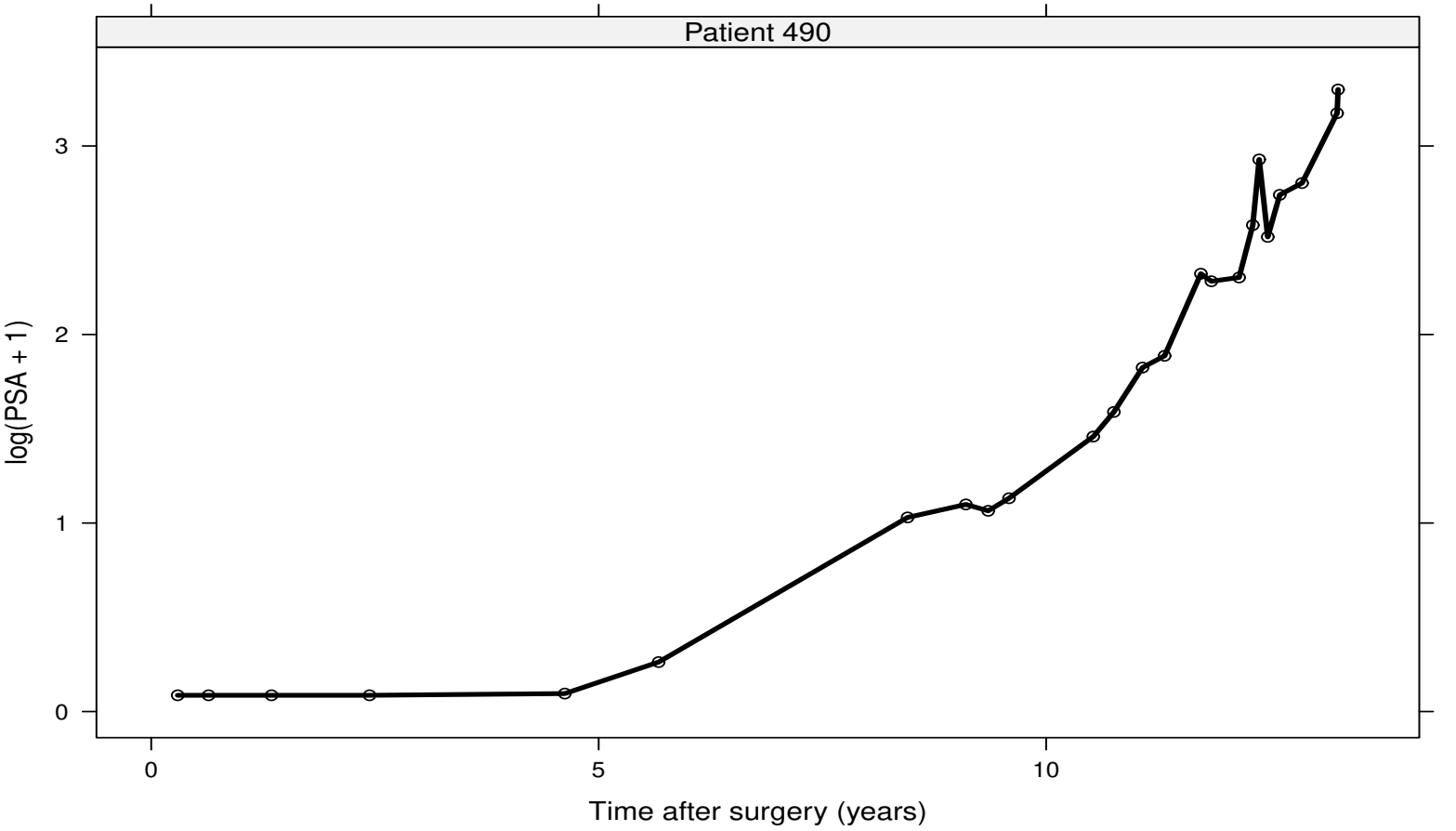
$$\Pr\{T_i^* \leq u \mid T_i^* > t, \mathcal{Y}_i(t), \mathcal{D}_n\} = 1 - \int \Pr\{T_i^* \geq u \mid T_i^* > t, \mathcal{Y}_i(t), \theta\}\, p(\theta \mid \mathcal{D}_n)\, \mathrm{d}\theta$$

- With the first term taking the form

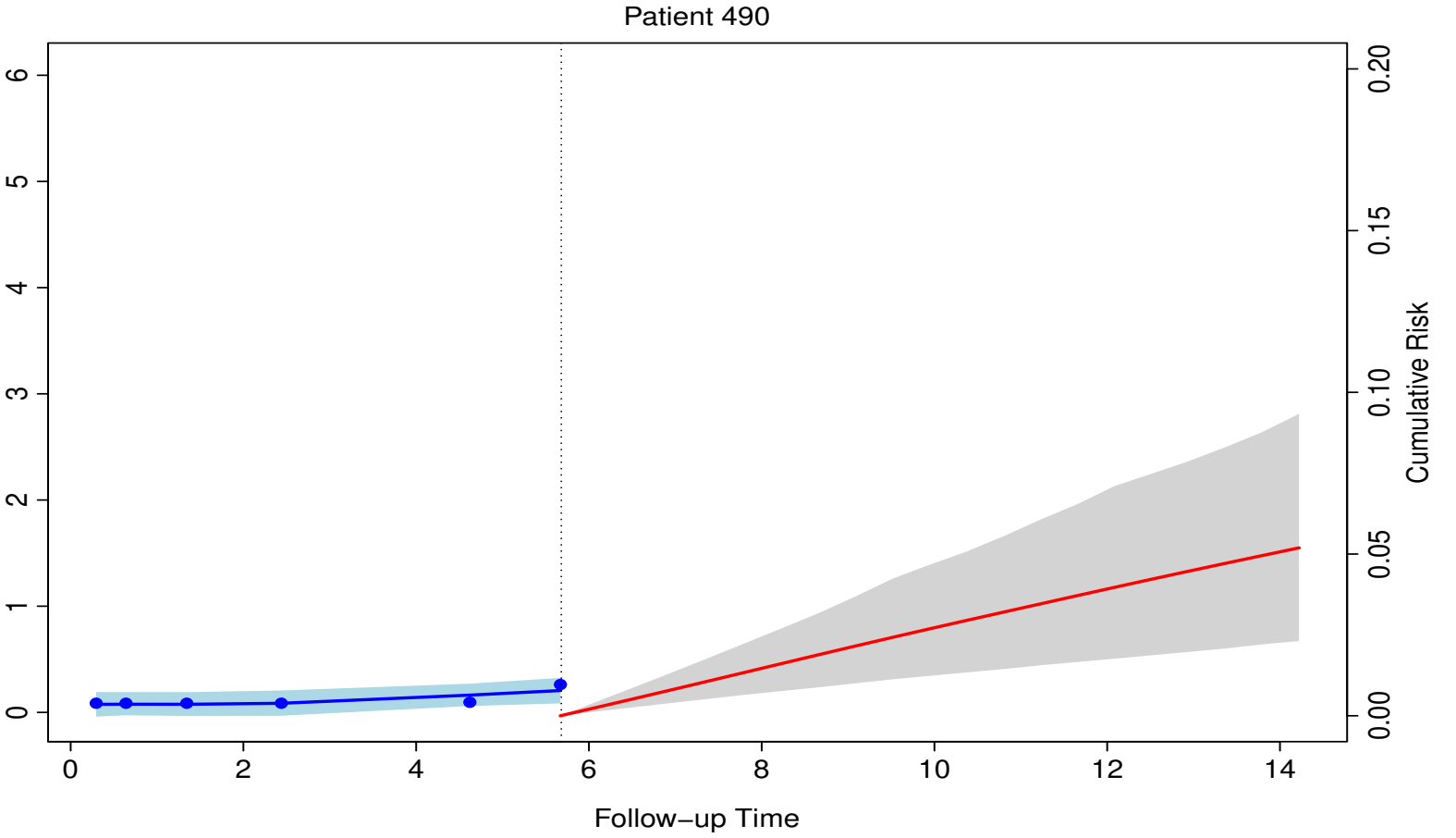
$$\Pr\{T_i^* \geq u \mid T_i^* > t, \mathcal{Y}_i(t), \theta\} =$$

$$= \int \frac{S_i\{u \mid \mathcal{H}_i(u, b_i, \theta), \theta\}}{S_i\{t \mid \mathcal{H}_i(t, b_i, \theta), \theta\}}\, p(b_i \mid T_i^* > t, \mathcal{Y}_i(t), \theta)\, \mathrm{d}b_i$$

# 2 Joint Models (cont'd)



# 2 Joint Models (cont'd)



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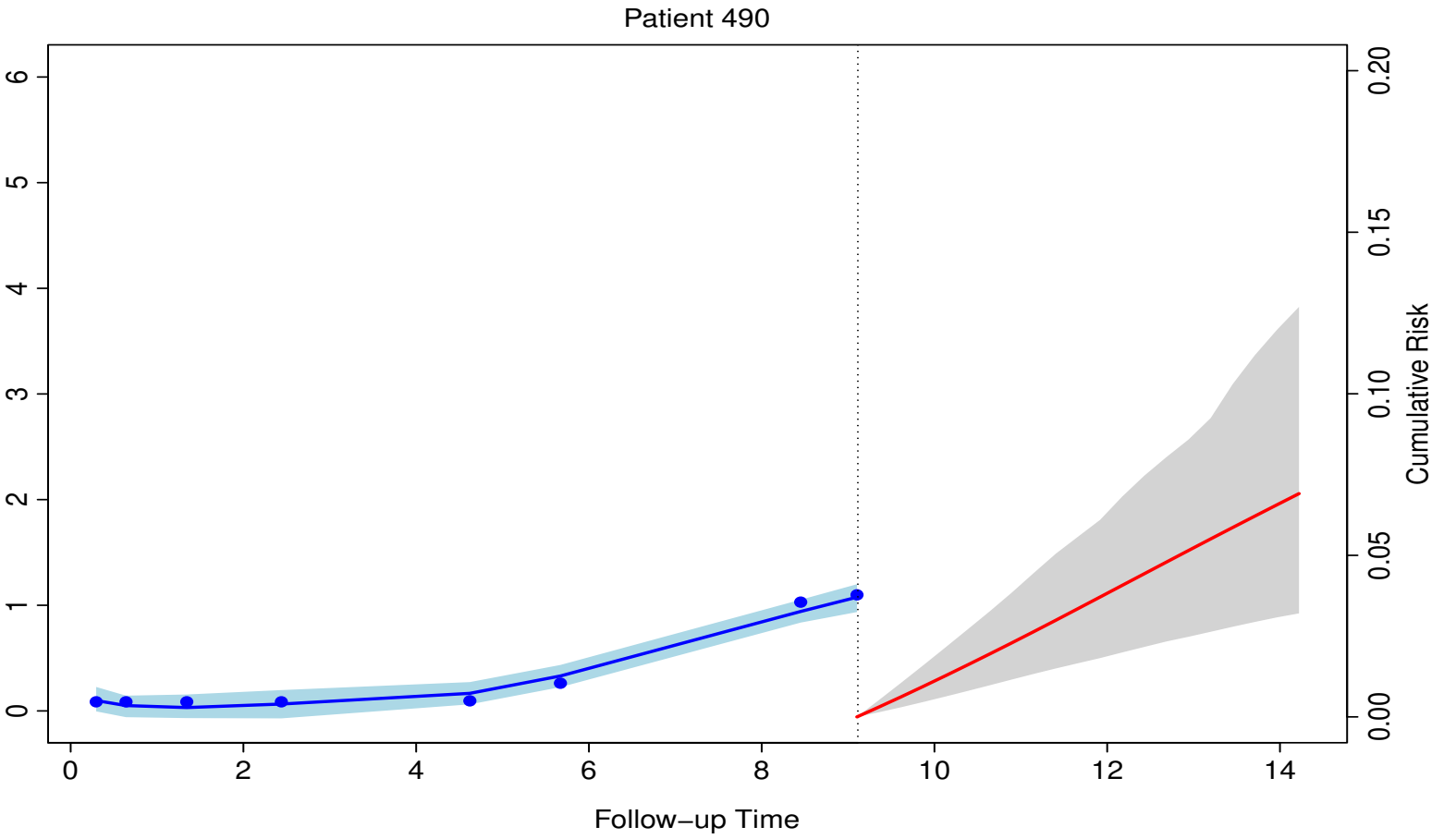

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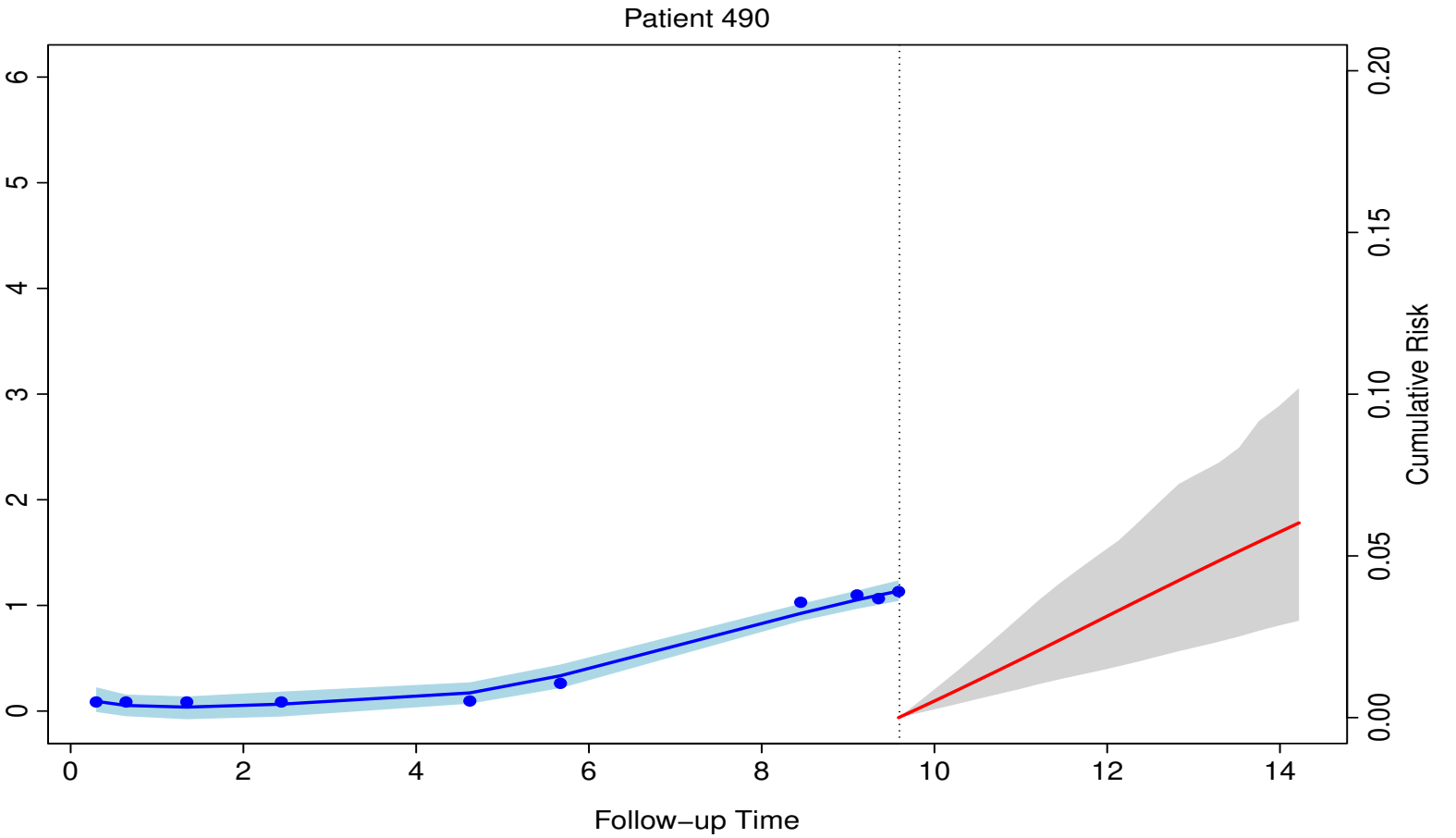

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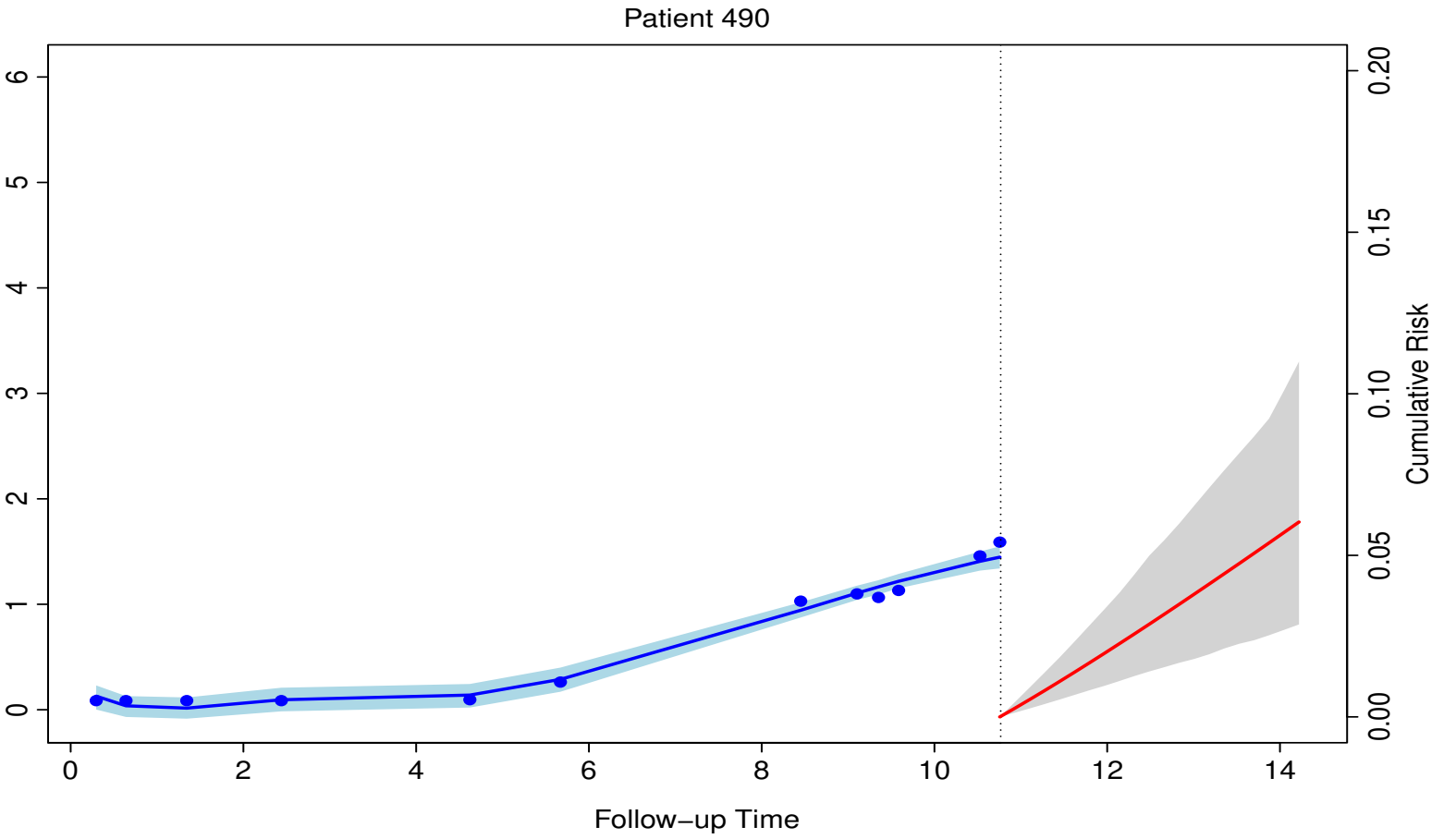

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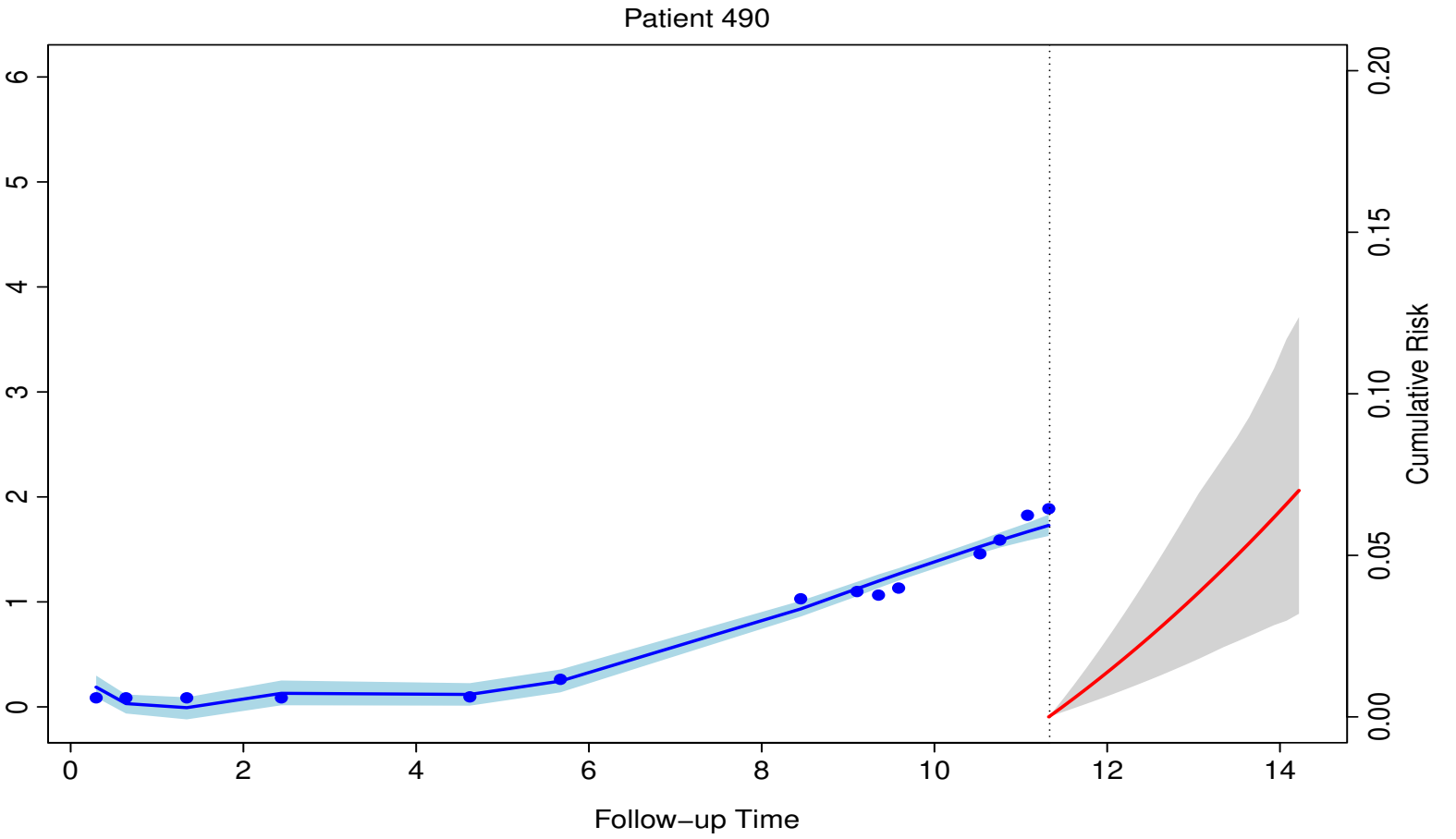

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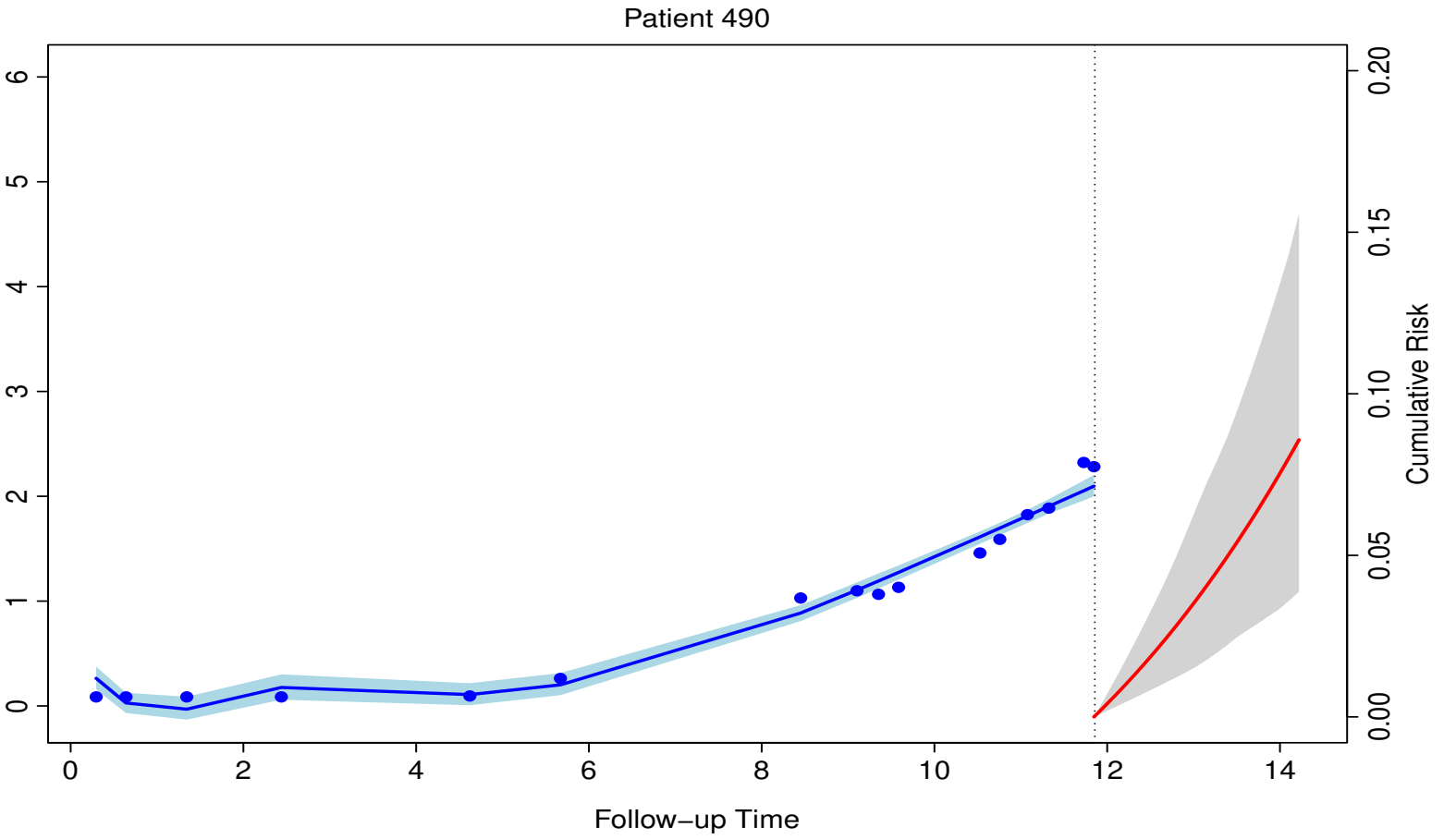

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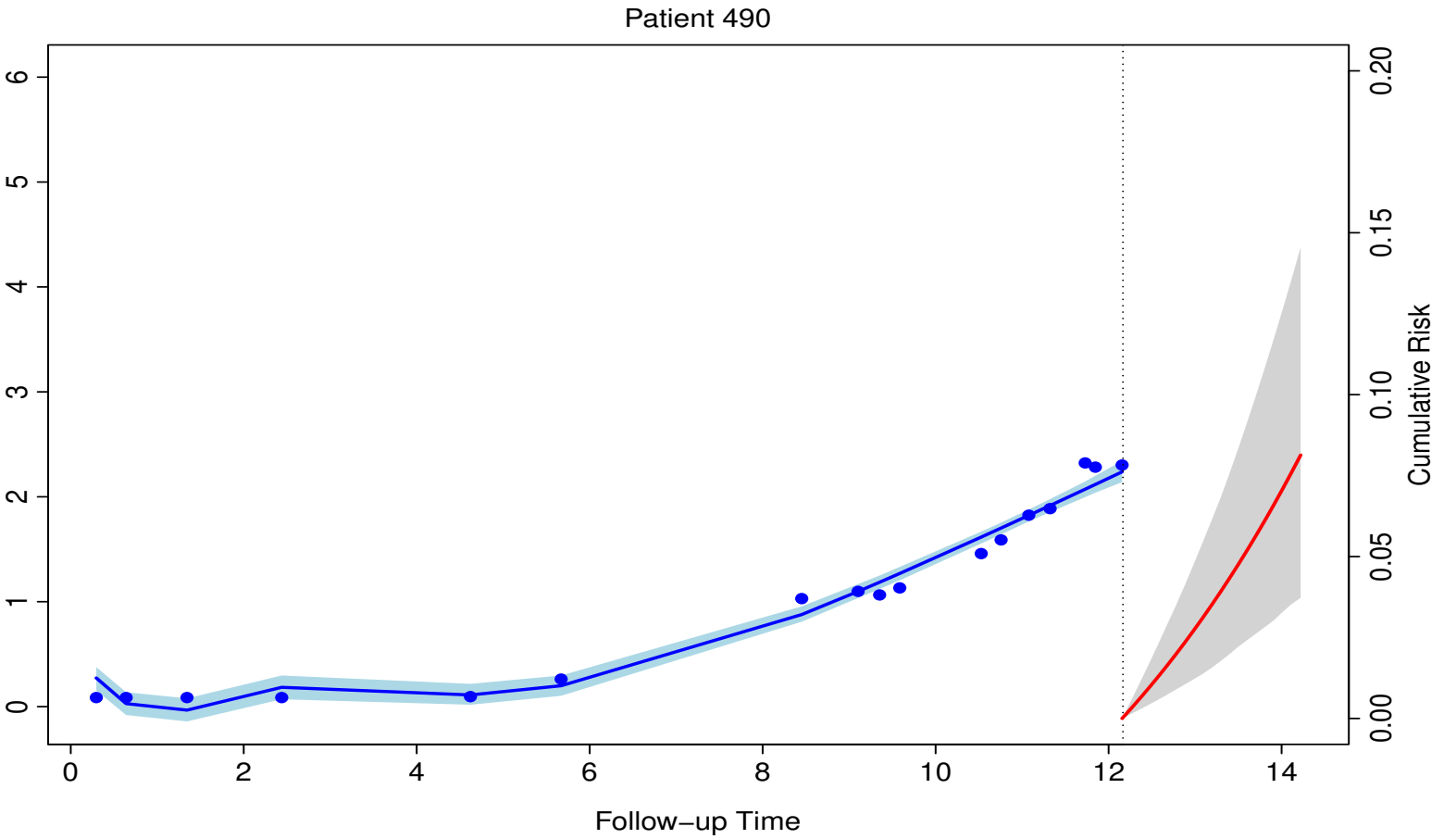

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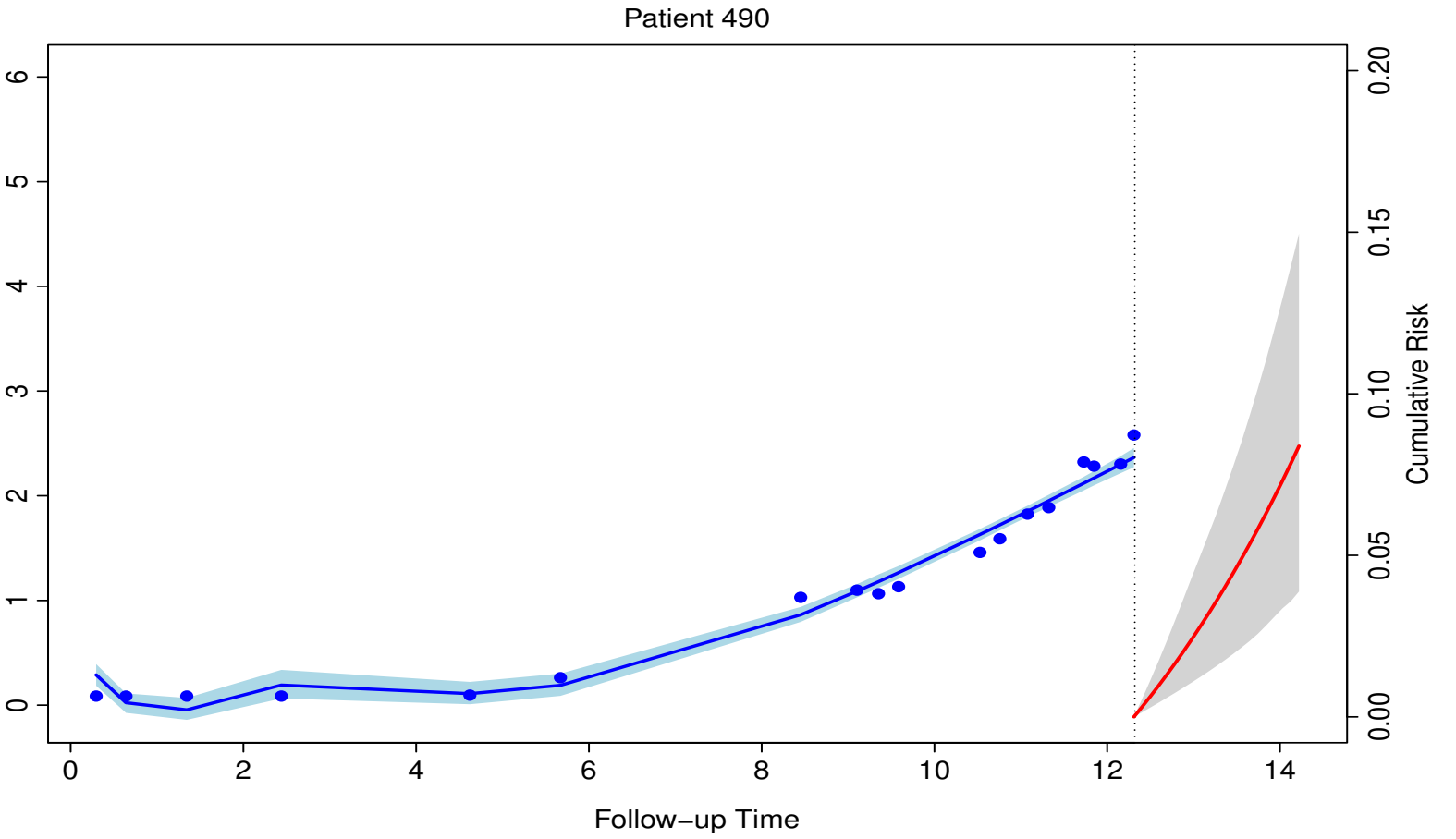

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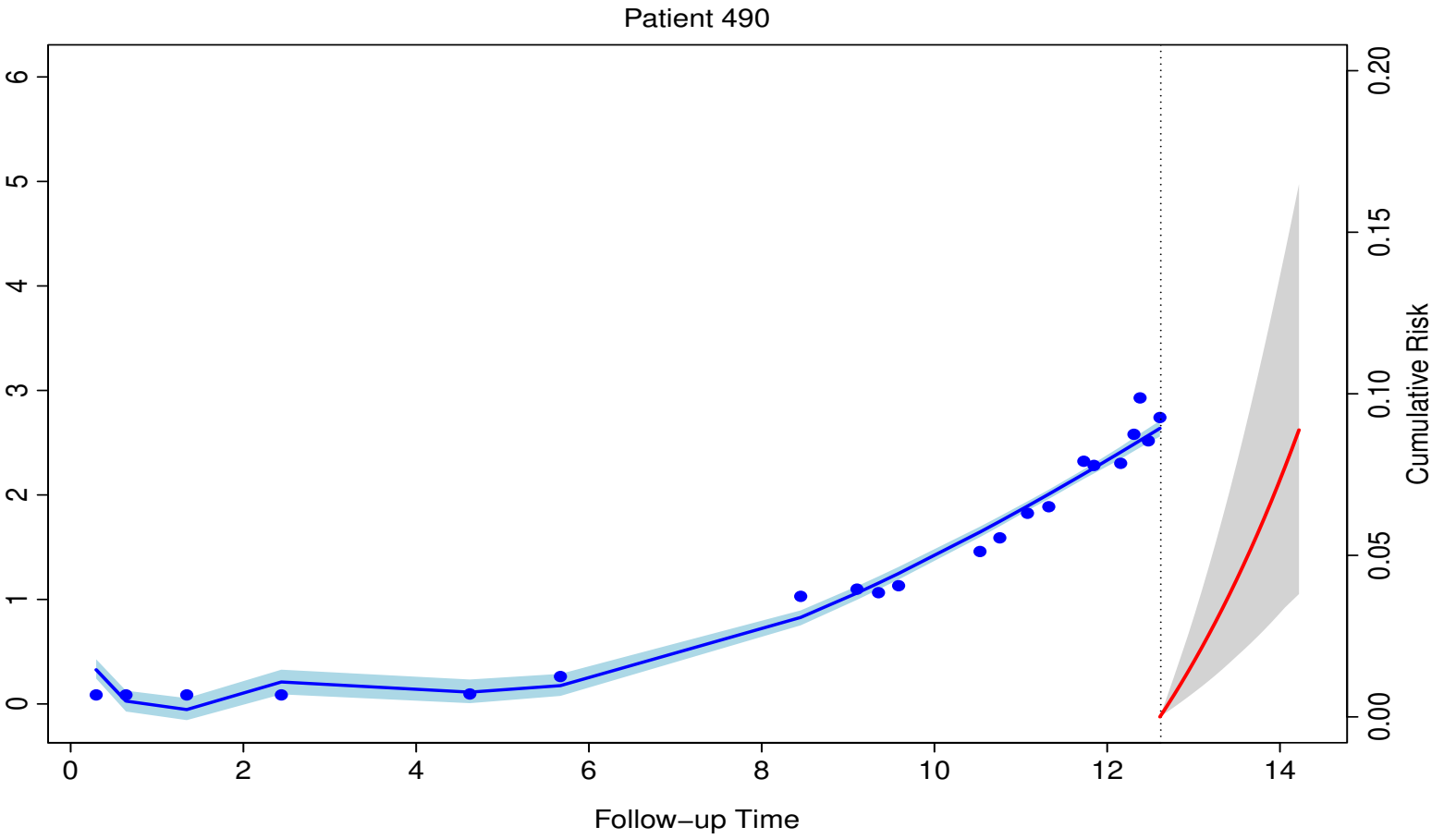

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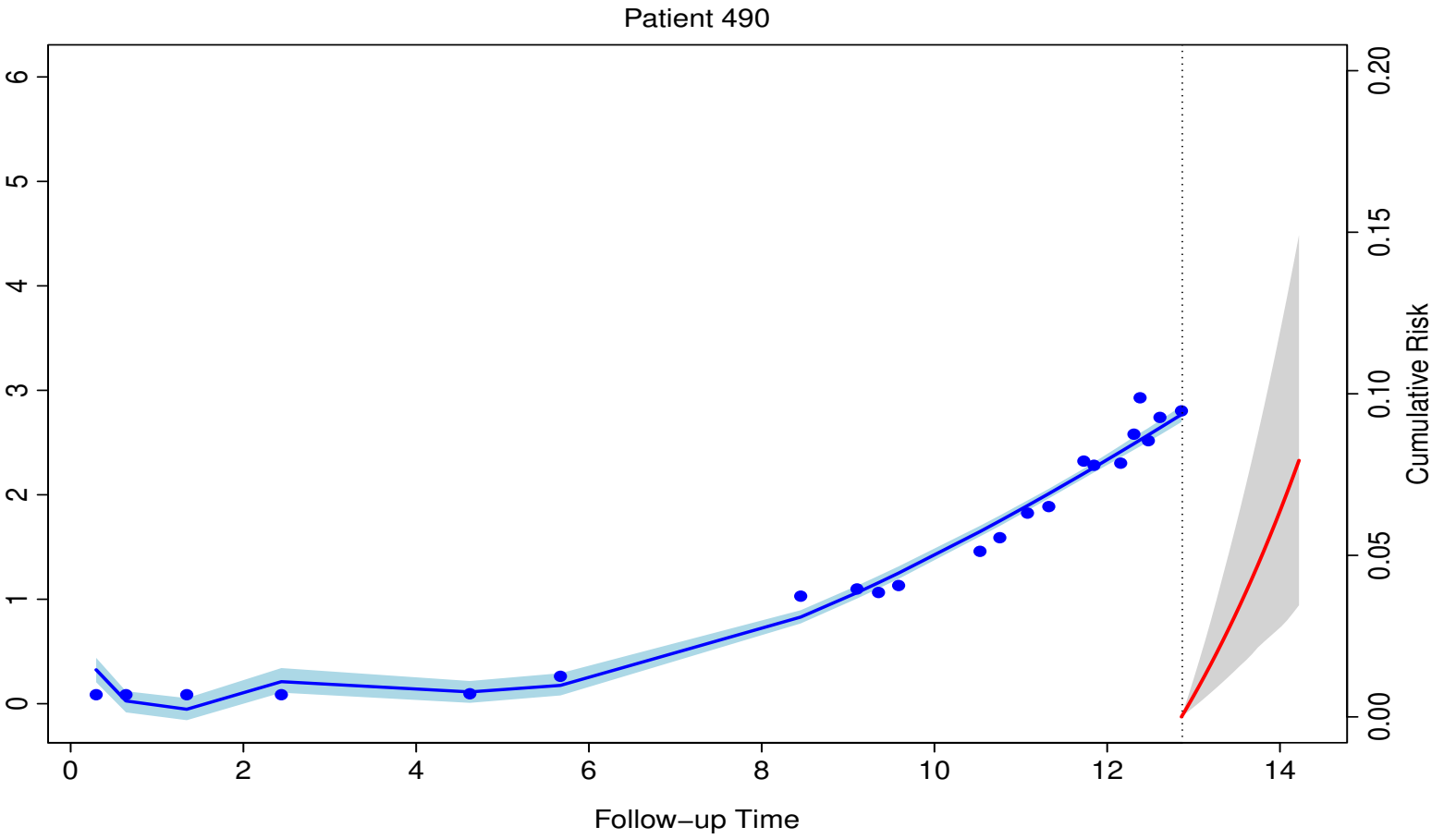
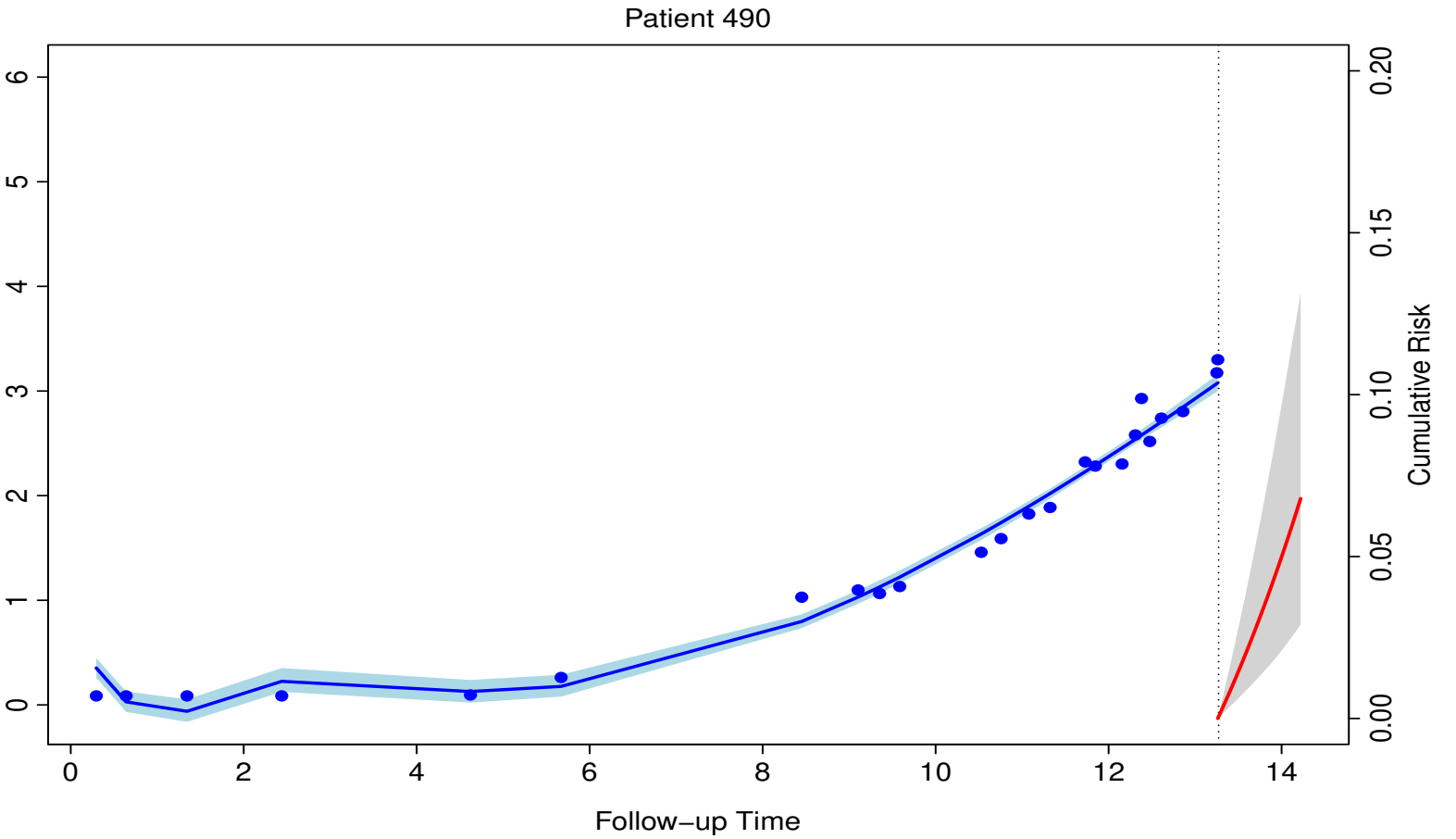


# 2 Joint Models (cont'd)



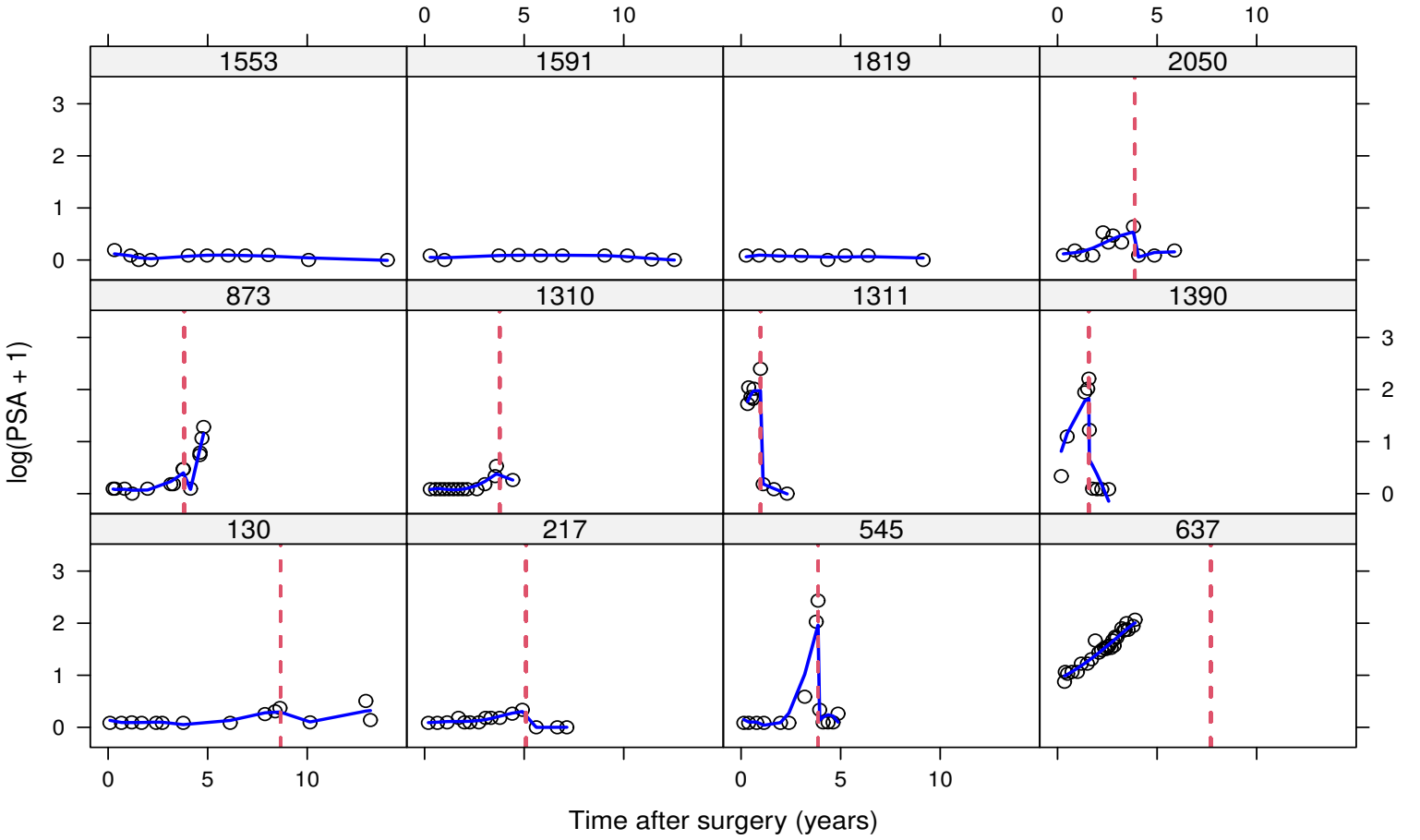
# 2 Joint Models (cont'd)

- In the context of dynamic predictions,
  - ▷ previous research has shown that predictive accuracy is compromised
  - ▷ when the model does not *adequately* capture the subject-specific trajectories shape

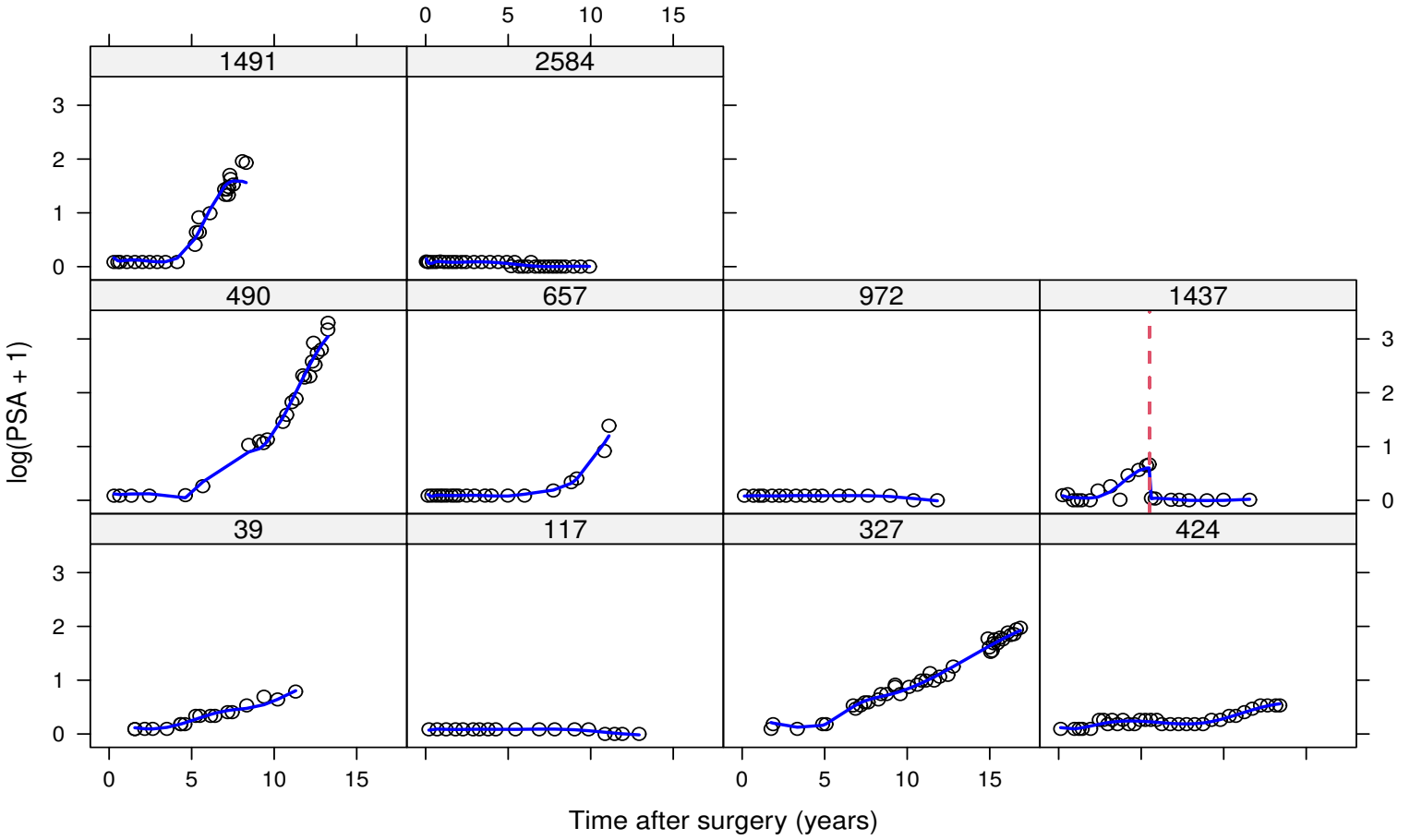
**Advice**

- ▷ use flexible models, e.g., splines in both fixed- and random-effects parts
- ▷ *increased computational burden*

# 2 Joint Models (cont'd)



# 2 Joint Models (cont'd)



# 3 Functional Forms

**There are different ways to link the longitudinal trajectories to the risk of an event**

▷ Some standard options are . . .

# 3 Functional Forms (cont'd)

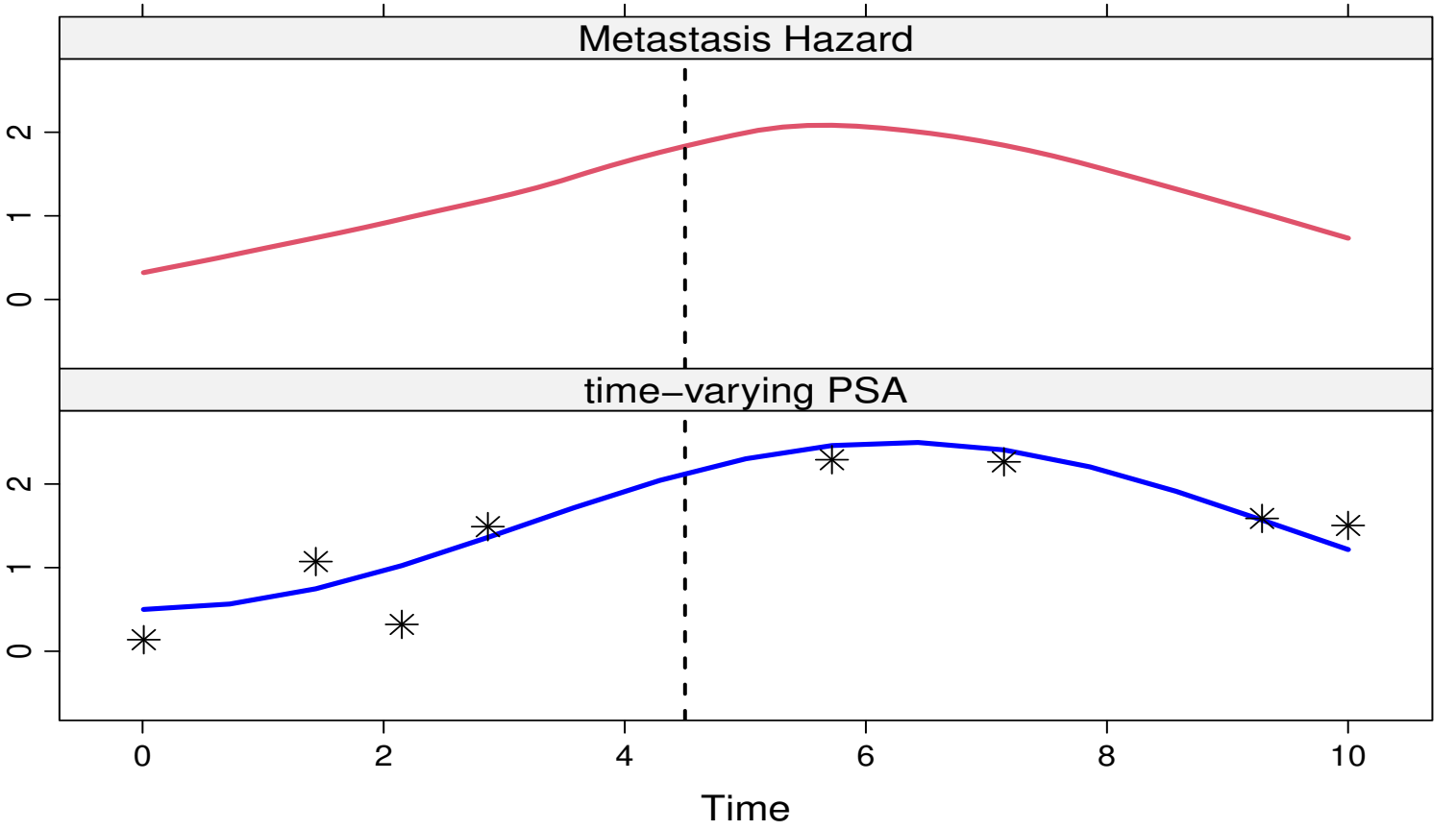
*Value:* The hazard of metastasis at $t$ is associated with the level of PSA at $t$:

$$h_i(t \mid \mathcal{H}_i(t, \boldsymbol{b}_i)) \;=\; h_0(t) \exp\{\boldsymbol{\gamma}^\top \boldsymbol{w}_i + \alpha \eta_i(t, \boldsymbol{b}_i)\}$$

where

$$\eta_i(t, \boldsymbol{b}_i) = \boldsymbol{x}_i^\top(t)\boldsymbol{\beta} + \boldsymbol{z}_i^\top(t)\boldsymbol{b}_i$$

# 3 Functional Forms (cont'd)



# 3 Functional Forms (cont'd)

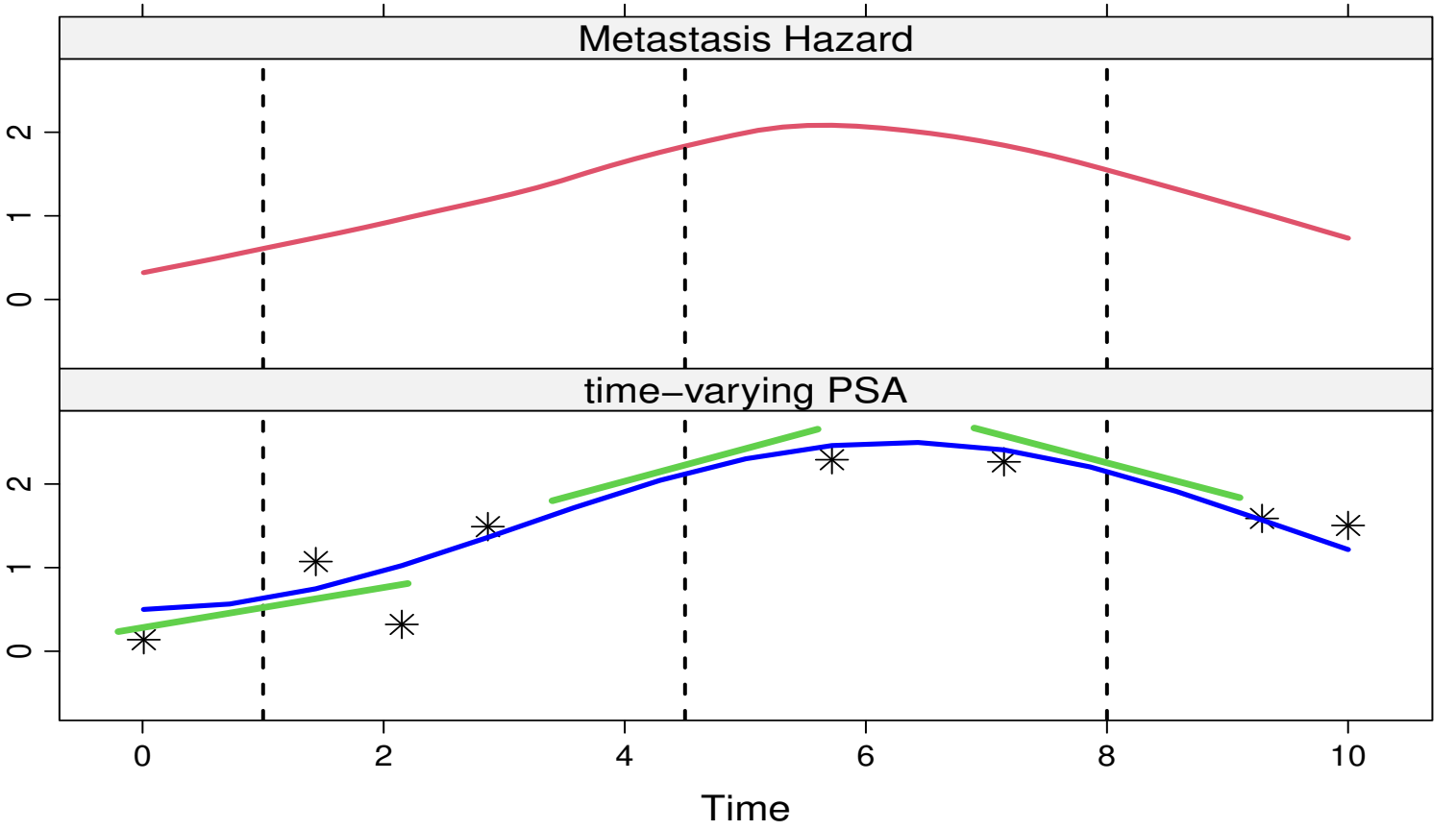
*Velocity:* The hazard of metastasis at $t$ is associated with the slope of the PSA trajectory at $t$:

$$h_i(t \mid \mathcal{H}_i(t, \boldsymbol{b}_i)) \;=\; h_0(t) \exp\{\boldsymbol{\gamma}^\top \boldsymbol{w}_i + \alpha \eta_i'(t, \boldsymbol{b}_i)\},$$

where

$$\eta_i'(t, \boldsymbol{b}_i) = \frac{\mathsf{d}}{\mathsf{d}t}\{\boldsymbol{x}_i^\top(t)\boldsymbol{\beta} + \boldsymbol{z}_i^\top(t)\boldsymbol{b}_i\}$$

# 3 Functional Forms (cont'd)



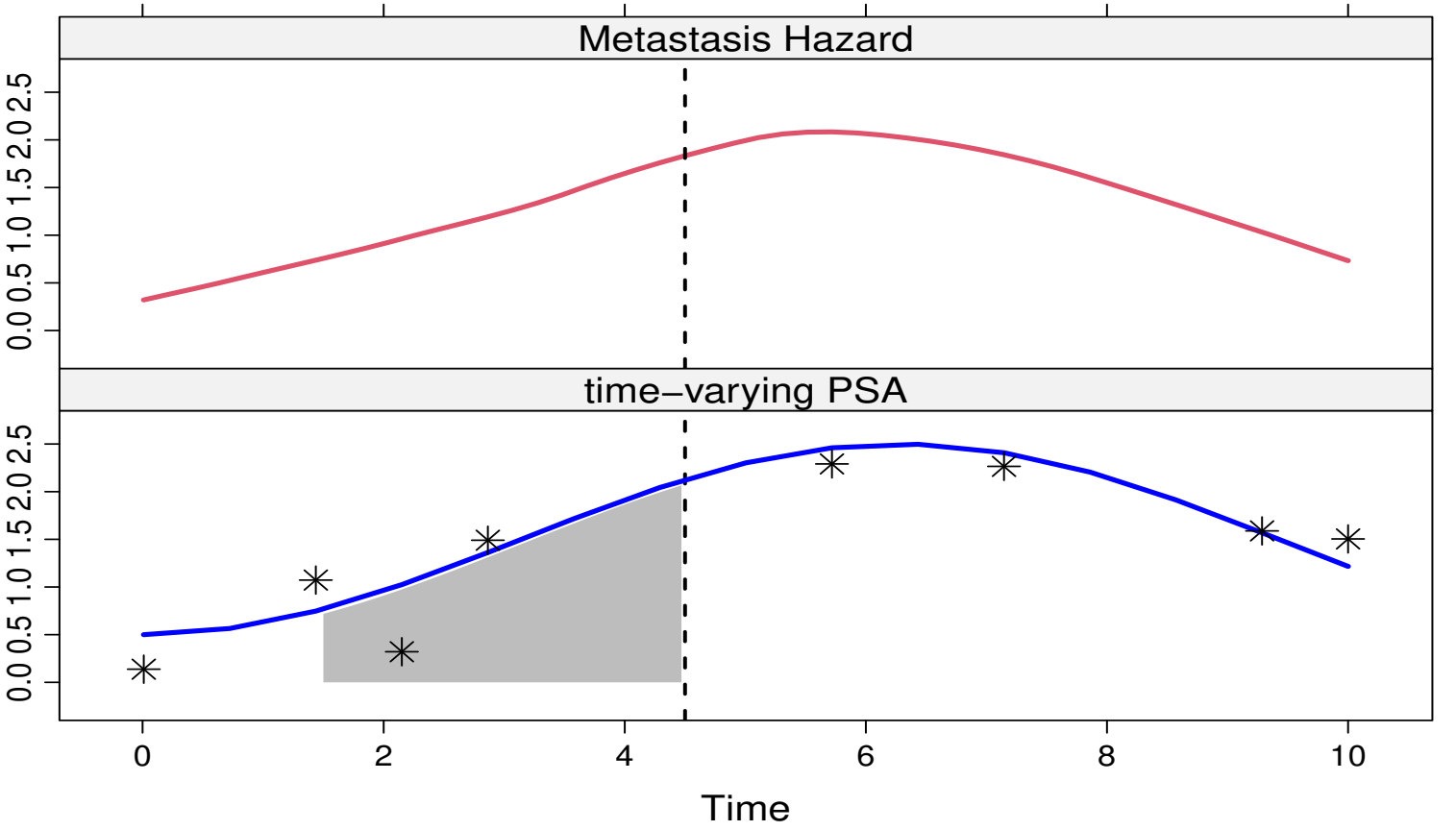
# 3 Functional Forms (cont'd)

*Average Effects:* The hazard of metastasis at $t$ is associated with the average PSA in the interval $(t - \Delta t, t)$:

$$h_i(t \mid \mathcal{H}_i(t, \boldsymbol{b}_i)) \;=\; h_0(t) \exp\left\{ \boldsymbol{\gamma}^\top \boldsymbol{w}_i + \alpha \frac{1}{\Delta t} \int_{t-\Delta t}^{t} \eta_i(s, \boldsymbol{b}_i) \; \mathrm{d}s \right\}$$

We account for the observation period

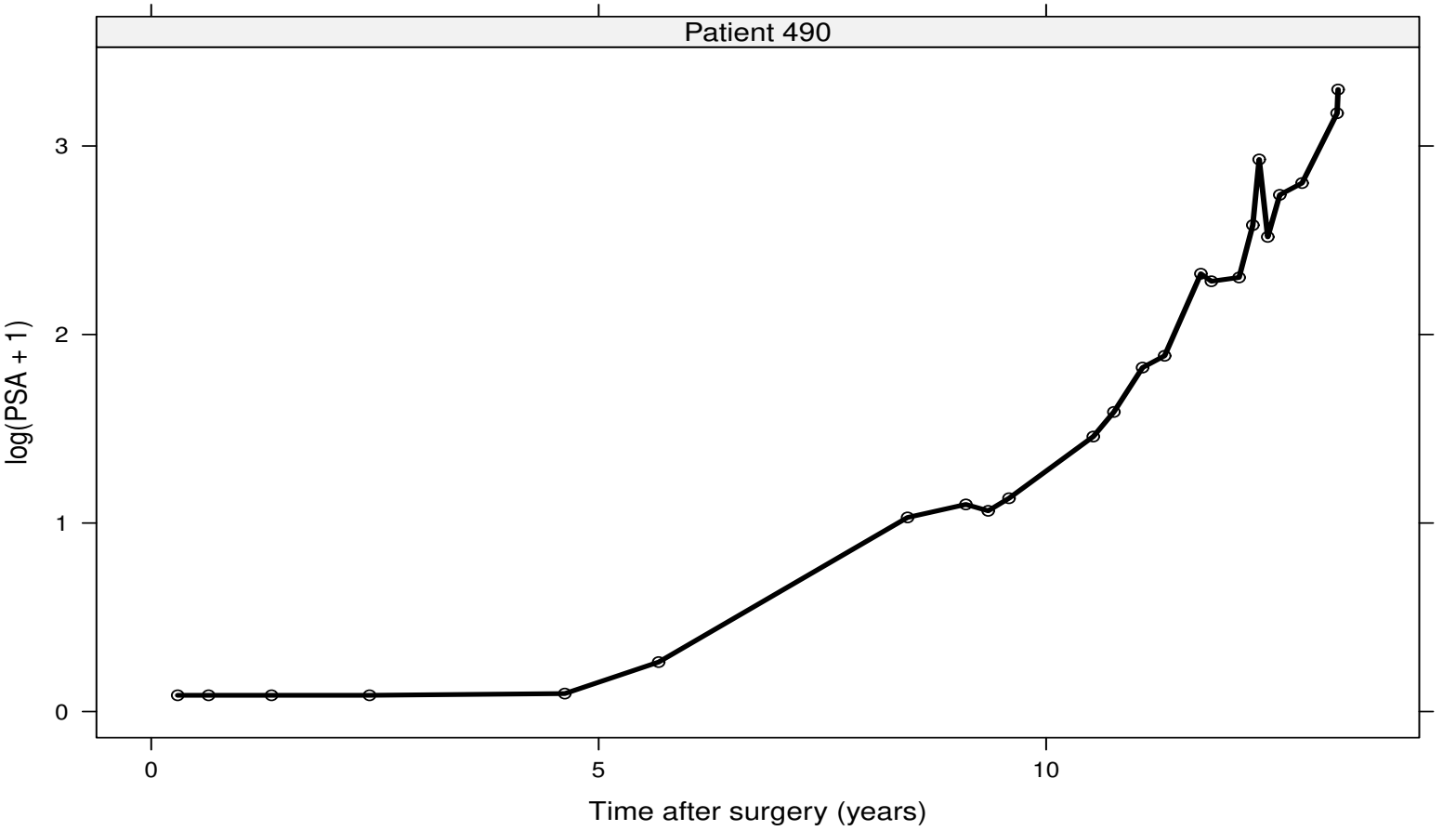
# 3 Functional Forms (cont'd)



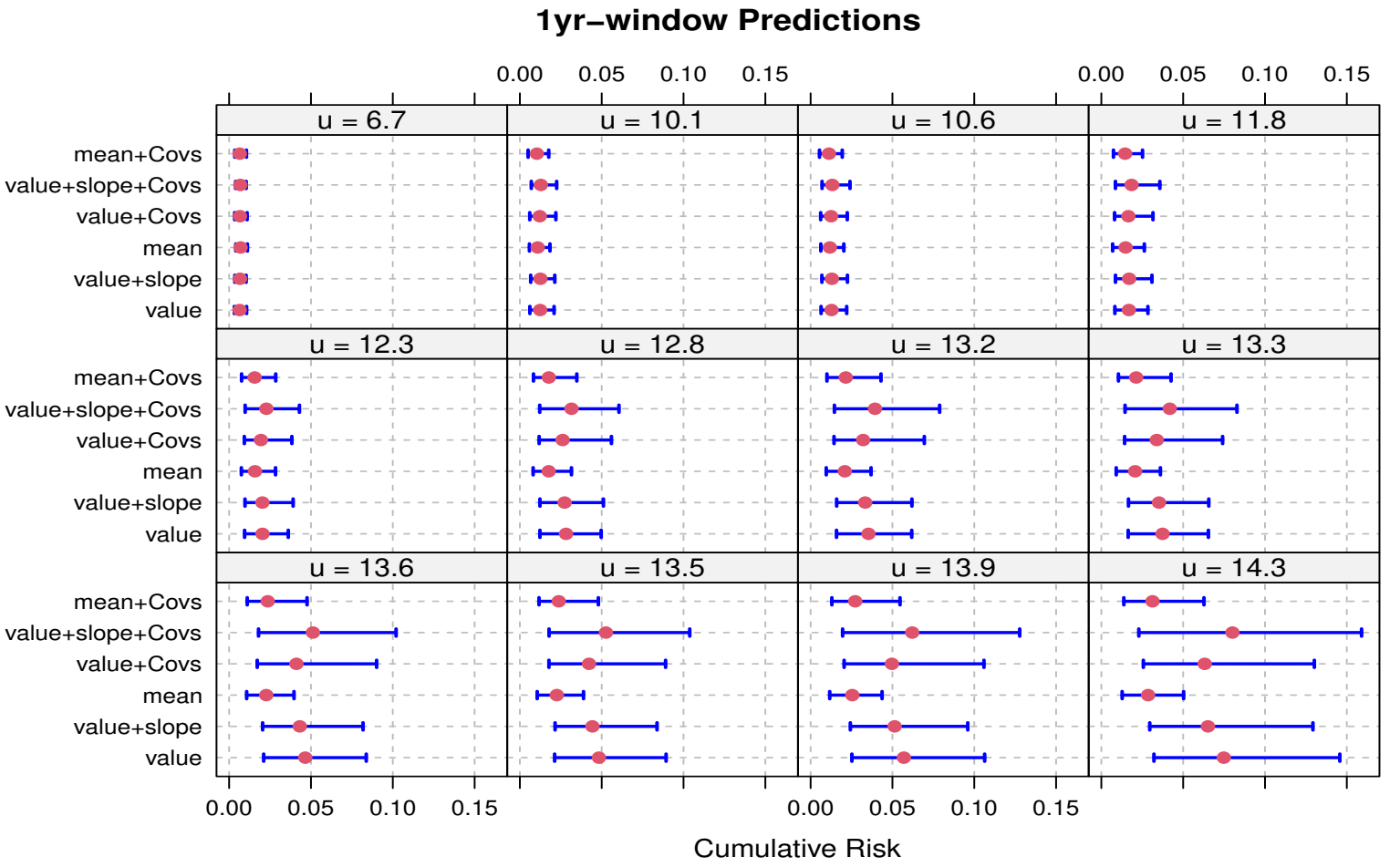
# 3 Functional Forms (cont'd)

**How significant is the choice of the functional form for dynamic predictions?**

# 3 Functional Forms (cont'd)



# 3 Functional Forms (cont'd)



# 4 Super Learning

- The selected functional form and time effect for the longitudinal outcome can influence the derived predictions
  - ▷ especially for the survival outcome

**How to select between the different functional forms and trajectory shapes?**

# 4 Super Learning (cont'd)

- The standard answer is to employ information criteria, e.g., DIC, WAIC, ...

- However, the longitudinal information dominates the joint likelihood
  $\Rightarrow$ will not be sensitive enough wrt predicting survival probabilities

- In addition, will *a single model* be the most appropriate

  ▷ for all follow-up times?

# 4 Super Learning (cont'd)

Solution

- ▷ *Consider multiple plausible models with different*
  - * longitudinal outcomes
  - * assumptions for the longitudinal profiles
  - * functional forms
  - * baseline covariates, interaction terms
  - * ...

- ▷ *Obtain the desired predictions from these models*

- ▷ *Combine predictions using weights*
  - * *how to select the weights?*

# 4 Super Learning (cont'd)

- Previous research: *Bayesian Model Averaging*
  - ▷ Assume we have a library of $L$ models $\mathcal{L} = \{M_1, \ldots, M_L\}$
  - ▷ Weights: Posterior probability of a model given the data

$$p(M_l \mid \mathcal{D}_n), \quad l = 1, \ldots, L$$

where

* $\mathcal{D}_n = \{T_i, \delta_i, \boldsymbol{y}_i; i = 1, \ldots, n\}$

# 4 Super Learning (cont'd)

- *Issues with BMA weights*
  - ▷ Requires calculating the marginal likelihood

$$p(\mathcal{D}_n \mid M_l) = \int \underbrace{p(\mathcal{D}_n \mid \boldsymbol{\theta}, M_l)}_{\text{Likelihood}} \underbrace{p(\boldsymbol{\theta} \mid M_l)}_{\text{Prior}} \, \mathrm{d}\boldsymbol{\theta}$$

  $\Rightarrow$ *Computationally demanding*

  - ▷ Weights not designed to optimize predictions
  - ▷ Not clear if we account for over-fitting

# 4 Super Learning (cont'd)

- *Issues with BMA weights*
  - ▷ The likelihood of a model that fits the data a bit better can have a likelihood value that is several units larger compared to the other models
  - ▷ Often one model dominates the weights over the others

# 4 Super Learning (cont'd)

Alternative Solution: *Super Learning*

- ▷ Select weights to optimize prediction metric *of your choice*
- ▷ Account for over-fitting using cross-validation

# 4 Super Learning (cont'd)

How it works:

- ▷ Assume we have a library of $L$ *base-learners* (models) $\mathcal{L} = \{M_1, \ldots, M_L\}$
- ▷ Specify the landmark time $t$, and a relevant future time $u$, $u > t$
- ▷ Split $\mathcal{D}_n$ in $V$-folds
- ▷ For $v \in \{1, \ldots, V\}$, train the learners in library $\mathcal{L}$ using $\mathcal{D}_n^{(-v)}$

# 4 Super Learning (cont'd)

How it works:

▷ For the subjects in $\mathcal{D}_n^{(v)}$, not used when training the learner, calculate the predictions

$$\hat{\pi}_i^{(v)}(u \mid t, M_l) = \Pr\{T_i^* < u \mid T_i^* > t, \mathcal{Y}_i(t), M_l, \mathcal{D}_n^{(-v)}\}$$

do this for all $v = 1, \ldots, V$ to get the *cross-validated predictions*

# 4 Super Learning (cont'd)

How it works:

▷ We define the ensemble of *cross-validated predictions*

$$\hat{\bar{\pi}}_i^v(u \mid t) = \sum_{l=1}^{L} \varpi_l(t) \hat{\pi}_i^{(v)}(u \mid t, M_l), \quad v = 1, \ldots, V$$

* the weights depend on $t \Rightarrow$ *different weights at different follow-up times*

# 4 Super Learning (cont'd)

How it works:

- Select $\varpi_l(t)$ to optimize your *meta-learner* (predictive accuracy metric), e.g.,
  - Brier Score (*Proper scoring rule*)
  - Expected Predictive Cross-Entropy (*Proper scoring rule*)
  - AUC (*Not a proper scoring rule*)
  - ...

- Under the constraints
  - $\widehat{\varpi}_l(t) > 0$ for all $l = 1, \ldots, L$
  - $\sum_{l=1}^{L} \widehat{\varpi}_l(t) = 1$

# 5 UM Data Analysis

A library $\mathcal{L}$ with twelve joint models

- PSA models
  - ▷ $M_{l1}$: *linear* subject-specific time trends that change after salvage
  - ▷ $M_{l2}$: the same as $M_{l1}$ + covariates
  - ▷ $M_{l3}$: *nonlinear* subject-specific time trends that change after salvage
  - ▷ $M_{l4}$: the same as $M_{l3}$ + covariates
- Baseline covariates: age at surgery, Charlson's index, Gleason score, and baseline PSA

# 5 UM Data Analysis (cont'd)

A library $\mathcal{L}$ with twelve joint models

- Metastasis models
  - ▷ $M_{s1}$: value of $\log(\text{PSA} + 1)$
  - ▷ $M_{s2}$: velocity of $\log(\text{PSA} + 1)$
  - ▷ $M_{s3}$: average $\log(\text{PSA} + 1)$
- Time varying salvage therapy
- Baseline covariates: the same as in the PSA models

# 5 UM Data Analysis (cont'd)

- We evaluated predictive accuracy in two time intervals
  - ▷ $(4, 7]$: 2514 patients at risk; 28 metastasis
  - ▷ $(6, 9]$: 1914 patients at risk; 16 metastasis

- Metrics - meta learners
  - ▷ Integrated Brier Score
  - ▷ Expected Predictive Cross-Entropy

# 5 UM Data Analysis (cont'd)

## Meta-learners

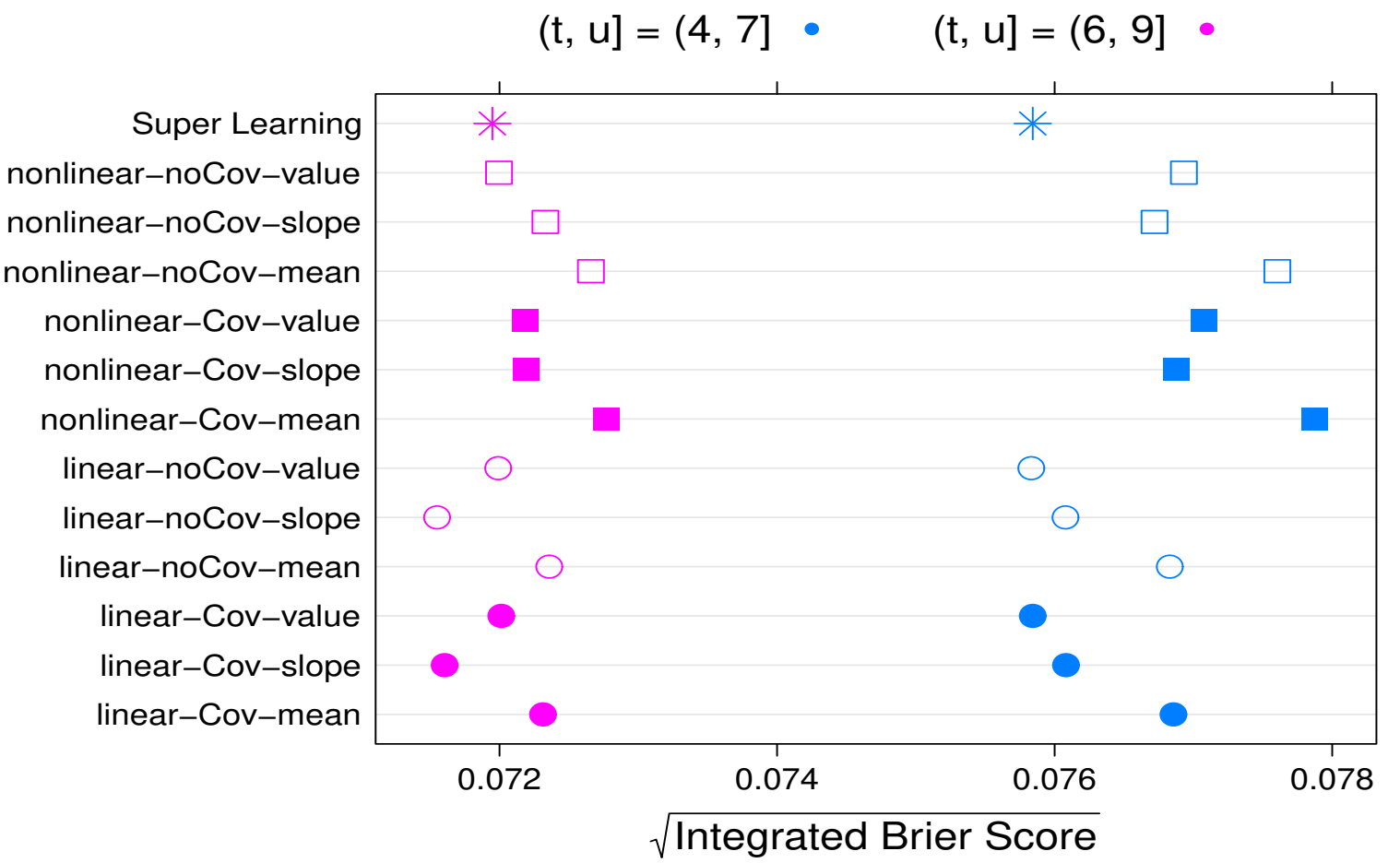
▷ *Integrated Brier Score*

$$\text{IBS}(u,t) = \frac{1}{u-t}\int_t^u E\Big\{\mathbb{I}(t < T_i^* \leq s) - \pi_i(s \mid t)\Big\}^2 \,\text{d}s$$

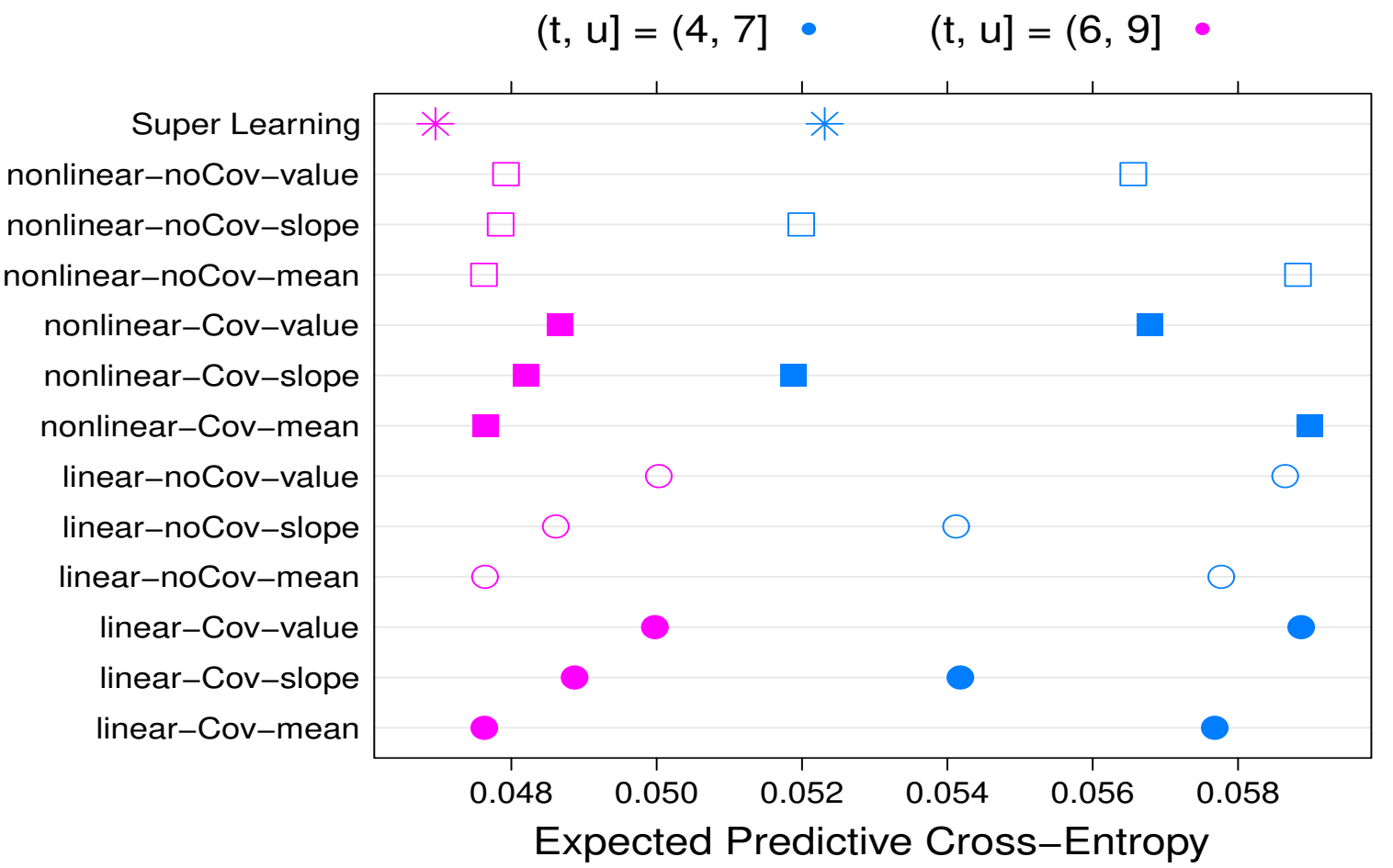
▷ *Expected Predictive Cross-Entropy*

$$\text{EPCE}(u,t) = E\Big\{-\log\Big[p\big\{T_i^* \mid t < T_i^* \leq u, \mathcal{Y}_i(t)\big\}\Big]\Big\}$$

# 5 UM Data Analysis (cont'd)



# 5 UM Data Analysis (cont'd)



# 5 UM Data Analysis (cont'd)

Observations (also from the simulation study)

- ▷ ensemble Super Learning (eSL) often, *but not always*, outperforms the individual models
- ▷ In some datasets and intervals $(t, u]$, the discrete Super Learner (dSL) beats the eSL

# 5 UM Data Analysis (cont'd)

Recommendation

**Regard eSL as an extra member of the library $\mathcal{L}$ and use CV to select the optimal strategy**

# 6 Software

- Available in **JMbayes2**
  - ▷ cross-validated fitting of models
  - ▷ combination of dynamic predictions
    https://drizopoulos.github.io/JMbayes2/articles/Super_Learning.html

**Thank for your attention!**

https://www.drizopoulos.com/

# 7 Choice of the Meta-Learner

We focus on two meta-learners

▷ *Integrated Brier Score*

$$\text{IBS}(t+\Delta t, t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} E\left[\left\{\mathbb{I}(T_i^* \leq s) - \pi_i(s \mid t)\right\}^2 \,\middle|\, T_i^* > t\right] \mathsf{d}s$$

▷ *Expected Predictive Cross-Entropy*

$$\text{EPCE}(t+\Delta t, t) = E\left\{-\log\left[p\left\{T_i^* \mid t < T_i^* \leq t+\Delta t, \mathcal{Y}_i(t)\right\}\right]\right\}$$

# 7 Choice of the Meta-Learner (cont'd)

- For the estimation of the Brier score, we need to account for censoring in $[t, t + \Delta t)$
  * inverse probability of censoring weighting
  * model-based weights

# 7 Choice of the Meta-Learner (cont'd)

- Brier Score with IPCW

$$\widehat{\mathsf{BS}}_{IPCW}(t+\Delta t,t)=\frac{1}{n}\sum_{i=1}^{n}\widehat{W}_i(t+\Delta t,t)\Big\{\mathbb{I}(T_i\leq t+\Delta t)-\hat{\hat{\pi}}_i^v(t+\Delta t\mid t)\Big\}^2$$

where

$$\widehat{W}_i(t+\Delta t,t)=\frac{\mathbb{I}(t<T_i\leq t+\Delta t)\delta_i}{\hat{G}(T_i\mid t)}+\frac{\mathbb{I}(T_i>t+\Delta t)}{\hat{G}(t+\Delta t\mid t)},$$

with $\hat{G}(\cdot)$ denoting Kaplan-Meier estimate of the censoring distribution $\Pr(C_i>t)$

# 7 Choice of the Meta-Learner (cont'd)

- Brier Score with model-weights

$$\widehat{\mathsf{BS}}_{model}(t+\Delta t,t)=\frac{1}{n_t}\sum_{i:T_i>t}\delta_i\mathbb{I}(T_i\leq t+\Delta t)\Big\{1-\hat{\tilde{\pi}}_i^v(t+\Delta t\mid t)\Big\}^2$$
$$+\,\mathbb{I}(T_i>t+\Delta t)\Big\{\hat{\tilde{\pi}}_i^v(t+\Delta t\mid t)\Big\}^2$$
$$+(1-\delta_i)\mathbb{I}(T_i\leq t+\Delta t)\Bigg[\hat{\tilde{\pi}}_i^v(t+\Delta t\mid T_i)\Big\{1-\hat{\tilde{\pi}}_i^v(t+\Delta t\mid t)\Big\}^2$$
$$+\Big\{1-\hat{\tilde{\pi}}_i^v(t+\Delta t\mid T_i)\Big\}\Big\{\hat{\tilde{\pi}}_i^v(t+\Delta t\mid t)\Big\}^2\Bigg]$$

# 7 Choice of the Meta-Learner (cont'd)

- IPCW
  - ▷ *Advantage:* it provides unbiased estimates even when the model is misspecified
  - ▷ *Disadvantage:* it requires that the model for the weights is correct
    * challenging because censoring may depend on the longitudinal outcomes in a complex manner
    * sensitive to (unobserved) instrument by confounder interactions

# 7 Choice of the Meta-Learner (cont'd)

- Model-based Weights
  - ▷ *Advantage:* it allows censoring to depend on the longitudinal history (in any possible manner)
  - ▷ *Disadvantage:* it requires that the model is well-specified

# 7 Choice of the Meta-Learner (cont'd)

- An estimate of EPCE$(t + \Delta t, t)$ that accounts for censoring

$$\widehat{\text{EPCE}}(t + \Delta t, t) = \frac{1}{n_t} \sum_{i:T_i > t} - \log\Big[p\{\tilde{T}_i, \tilde{\delta}_i \mid T_i > t, \mathcal{Y}_i(t), \mathcal{D}_n\}\Big]$$

  with

  ▷ $\tilde{T}_i = \min(T_i, t + \Delta t)$

  ▷ $\tilde{\delta}_i = \delta_i \mathbb{I}(t < T_i \leq t + \Delta t)$

- Features

  ▷ it allows censoring to depend on the longitudinal history

  ▷ *problem:* it is not written as a function of the predictions

# 7 Choice of the Meta-Learner (cont'd)

- The conditional predictive log-likelihood

$$\log\Big[p\{\tilde{T}_i, \tilde{\delta}_i \mid T_i > t, \mathcal{Y}_i(t), \mathcal{D}_n\}\Big] =$$
$$\tilde{\delta}_i \log[h_i\{\tilde{T}_i \mid \mathcal{Y}_i(t), \mathcal{D}_n\}] + \log \frac{\Pr\{T_i^* > \tilde{T}_i \mid \mathcal{Y}_i(t), \mathcal{D}_n\}}{\Pr\{T_i^* > t \mid \mathcal{Y}_i(t), \mathcal{D}_n\}}$$

  - ▷ the second term is $\log\{\pi_i(\tilde{T}_i \mid t)\}$
  - ▷ for the first term, we write the hazard function as

$$h_i\{\tilde{T}_i \mid \mathcal{Y}_i(t), \mathcal{D}_n\} = \frac{p(\tilde{T}_i)}{S(\tilde{T}_i)} = -\frac{\frac{\mathrm{d}}{\mathrm{d}t}\Pr\{T_i^* > t \mid \mathcal{Y}_i(t), \mathcal{D}_n\}\Big|_{t=\tilde{T}_i}}{\Pr\{T_i^* > \tilde{T}_i \mid \mathcal{Y}_i(t), \mathcal{D}_n\}}$$

# 7 Choice of the Meta-Learner (cont'd)

- We approximate the derivative with a forward difference and we get

$$\widehat{\mathsf{EPCE}}(t+\Delta t, t) =$$

$$-\frac{1}{n_t}\sum_{i:T_i>t}\tilde{\delta}_i\left[\log\{1-\hat{\tilde{\pi}}_i^v(\tilde{T}_i+\epsilon \mid \tilde{T}_i)\}-\log(\epsilon)\right]+\log\{\hat{\tilde{\pi}}_i^v(\tilde{T}_i \mid t)\}$$

that can be used to optimize $\varpi_l(t)$

# 7 UM Data Analysis (cont'd)

| | $(t, t+\Delta t] = (4, 7]$ | | $(t, t+\Delta t] = (6, 9]$ | |
|---|---|---|---|---|
| | IBS | weights | IBS | weights |
| SL | 0.07584 | | 0.07195 | |
| linear-noCov-value | 0.07583 | 0.00000 | 0.07199 | 0.08333 |
| linear-noCov-slope | 0.07608 | 0.00000 | 0.07155 | 0.08340 |
| linear-noCov-mean | 0.07683 | 0.00000 | 0.07236 | 0.08332 |
| linear-Cov-value | 0.07584 | 1.00000 | 0.07201 | 0.08335 |
| linear-Cov-slope | 0.07608 | 0.00000 | 0.07160 | 0.08339 |
| linear-Cov-mean | 0.07686 | 0.00000 | 0.07231 | 0.08332 |
| nonlinear-noCov-value | 0.07693 | 0.00000 | 0.07200 | 0.08334 |
| nonlinear-noCov-slope | 0.07672 | 0.00000 | 0.07233 | 0.08331 |
| nonlinear-noCov-mean | 0.07760 | 0.00000 | 0.07266 | 0.08329 |
| nonlinear-Cov-value | 0.07708 | 0.00000 | 0.07218 | 0.08332 |
| nonlinear-Cov-slope | 0.07687 | 0.00000 | 0.07219 | 0.08333 |
| nonlinear-Cov-mean | 0.07788 | 0.00000 | 0.07277 | 0.08328 |

# 7 UM Data Analysis (cont'd)

| | $(t, t+\Delta t] = (4, 7]$ | | $(t, t+\Delta t] = (6, 9]$ | |
|---|---|---|---|---|
| | EPCE | weights | EPCE | weights |
| SL | 0.05231 | | 0.04696 | |
| linear-noCov-value | 0.05865 | 0.08325 | 0.05003 | 0.00002 |
| linear-noCov-slope | 0.05412 | 0.08320 | 0.04861 | 0.00000 |
| linear-noCov-mean | 0.05777 | 0.08260 | 0.04764 | 0.39649 |
| linear-Cov-value | 0.05887 | 0.08215 | 0.04997 | 0.00000 |
| linear-Cov-slope | 0.05418 | 0.08333 | 0.04887 | 0.00000 |
| linear-Cov-mean | 0.05768 | 0.08270 | 0.04763 | 0.12793 |
| nonlinear-noCov-value | 0.05656 | 0.08337 | 0.04793 | 0.00136 |
| nonlinear-noCov-slope | 0.05199 | 0.08517 | 0.04785 | 0.44966 |
| nonlinear-noCov-mean | 0.05882 | 0.08296 | 0.04762 | 0.00961 |
| nonlinear-Cov-value | 0.05679 | 0.08315 | 0.04867 | 0.00000 |
| nonlinear-Cov-slope | 0.05188 | 0.08526 | 0.04820 | 0.01327 |
| nonlinear-Cov-mean | 0.05899 | 0.08288 | 0.04764 | 0.00166 |